Numerical shockwave anomalies in presence of hydraulic jumps in the SWE with variable bed elevation

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A. Navas-Montilla, J. Murillo
Fluid Mechanics. EINA, University of Zaragoza
anavas@unizar.es, Javier.Murillo@unizar.es

Abstract

When solving the shallow water equations appropriate numerical solvers must allow energy-dissipative solutions in presence of steady and unsteady hydraulic jumps. Hydraulic jumps are present in surface flows and may produce significant morphological changes. Unfortunately, it has been documented that some numerical anomalies may appear. These anomalies are the incorrect positioning of steady jumps and the presence of a spurious spike of discharge inside the cell containing the jump produced by a non-linearity of the Hugoniot locus connecting the states at both sides of the jump. Therefore, this problem remains unresolved in the context of Godunovs schemes applied to shallow flows. This issue is usually ignored as it does not affect to the solution in steady cases. However, it produces undesirable spurious oscillations in transient cases that can lead to misleading conclusions when moving to realistic scenarios. Using spike-reducing techniques based on the construction of interpolated fluxes, it is possible to define numerical methods including discontinuous topography that reduce the presence of the aforementioned numerical anomalies.

1. Introduction

It has been widely reported in the literature that anomalies arise in presence of shock waves. An example of such problems are the Carbuncle [1], the slowly-moving shock [2] and the wall-heating phenomenon [3], all of them leading to spurious numerical solutions. Here we focus on the slowly-moving shock problem:

▷ It was first investigated by Roberts in [2], who defined it as numerical noise generated in the discrete shock transition layer which is transported downstream (post-shock oscillations). It is depicted in Fig. 1.

▷ The slowly-moving shock problem has has its origin in the nonlinearity of the Hugoniot locus.

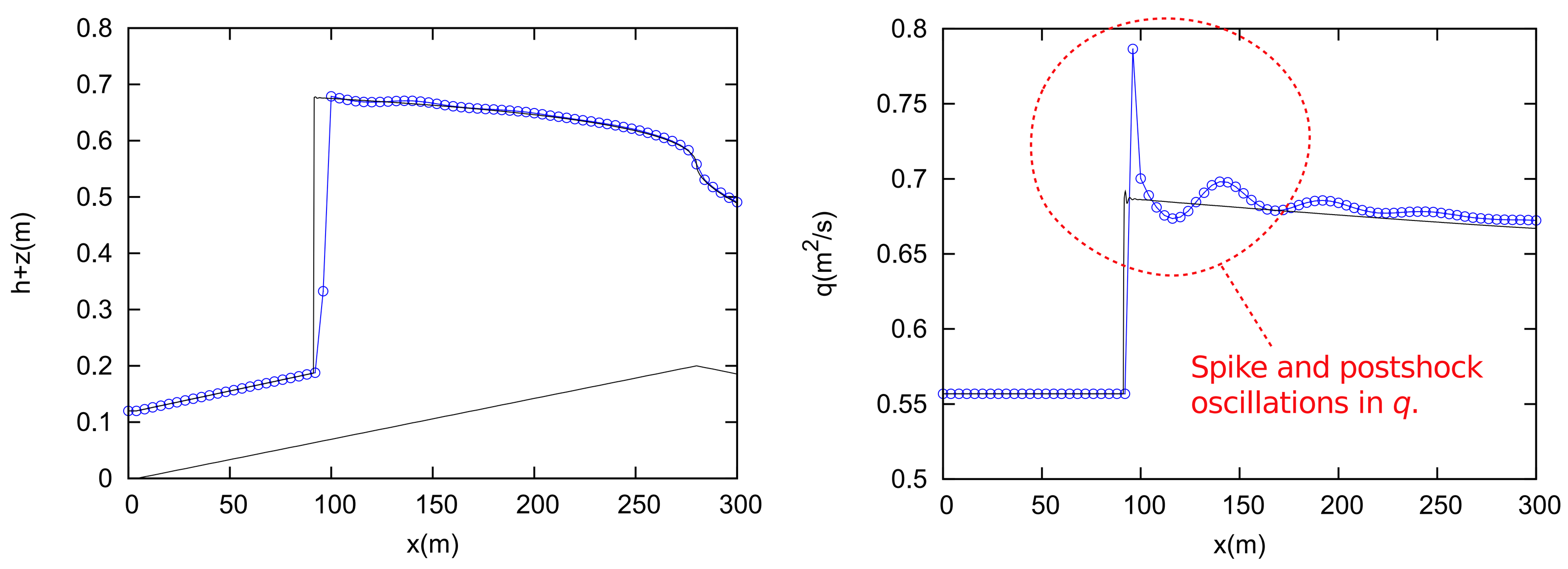


Figure 1: Numerical (blue) and analytical (black) solution for a slowly-moving shock (hydraulic jump) climbing an inclined plane.

The basic ideas underlying this work can be illustrated by examining hyperbolic nonlinear systems of equations with source terms in 1D

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}\,, \tag{1}$$

where $\mathbf{U} = \mathbf{U}(x,t) \in \mathcal{C} \subset \mathbb{R}^{N_\lambda}$ is the vector of conserved quantities, $\mathbf{F} = \mathbf{F}(\mathbf{U})$ is the flux function and $\mathbf{S}$ is the source term.

In this work, we focus on the Shallow Water Equations over varying bed. Using vector notation, the components in (1) read

$$\mathbf{U} = \begin{pmatrix} h \\ hu \end{pmatrix}, \; \mathbf{F} = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}, \; \mathbf{S} = \begin{pmatrix} 0 \\ S_z \end{pmatrix}, \tag{2}$$

where h is the water depth, u is the depth averaged velocity, $q = hu$ the discharge and g is the acceleration of gravity. The source term S_z involves the variations in bed geometry $S_z = -gh\frac{dz}{dx}$ where z stands for the bed elevation.

2. Discontinuous solutions in the SWE

Across a shock wave, $\mathbf{U}_L$ and $\mathbf{U}_R$ are connected by the RH condition. The Hugoniot locus associated to the m-th characteristic field is given by

$$\mathcal{H}^m = \left\{ \mathbf{U}_R(\xi) \in \mathcal{C} \subset \mathbb{R}^{N_\lambda} \,|\, \xi \geq 0 \right\}\,, \tag{3}$$

where the states $\mathbf{U}_R(\xi)$ satisfy the RH condition

$$\mathbf{F}(\mathbf{U}_R(\xi)) - \mathbf{F}(\mathbf{U}_L) = \mathcal{S}^m \left(\mathbf{U}_R(\xi) - \mathbf{U}_L\right) \tag{4}$$

2.1. Slowly-moving shocks in the SWE: hydraulic jumps

The Hugoniot locus for the 1-characteristic field (red line in Fig. 1) is non-monotone, allowing:

▷ Steady shocks: RP($\mathbf{U}_L, \mathbf{U}_{R,2}$), where

$$\mathcal{S} = \frac{(hu)_L - (hu)_{R,2}}{h_L - h_{R,2}} = 0 \quad \text{since } (hu)_L = (hu)_{R,2} \tag{5}$$

▷ Slow moving shocks: RP($\mathbf{U}_L, \mathbf{U}_{R,1}$), where

$$\mathcal{S} = \frac{(hu)_L - (hu)_{R,2}}{h_L - h_{R,2}} > 0 \tag{6}$$

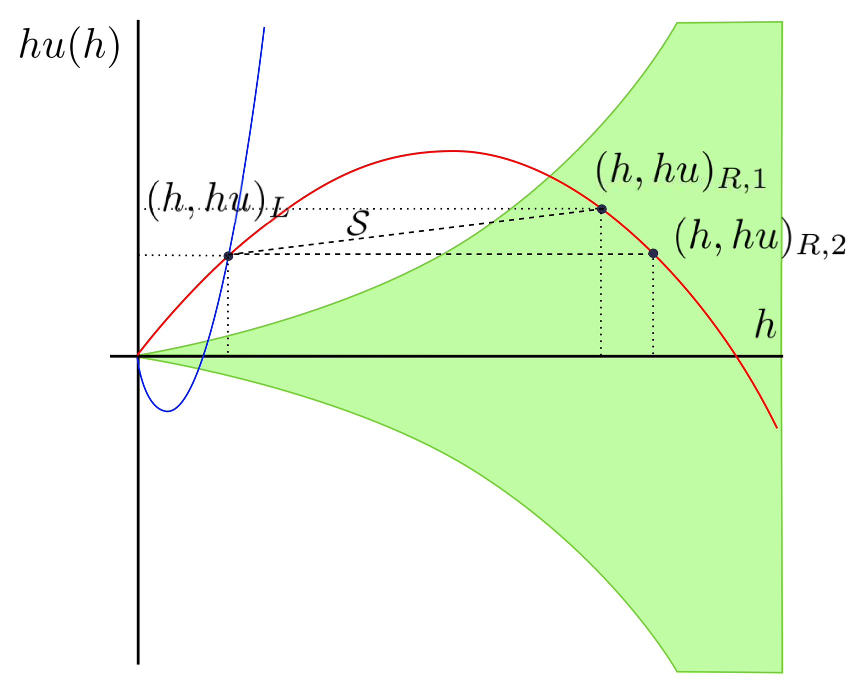


Figure 2: Hugoniot locus for a steady shock and for a slowly-moving shock.

3. Godunov's updating scheme and slowly-moving shocks

When using the approach proposed by Godunov, (1) is integrated inside $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times [t^n, t^{n+1}]$ leading to the following updating scheme

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{F}^-_{i+1/2} - \mathbf{F}^+_{i-1/2}]\,, \tag{7}$$

with $\mathbf{F}^-_{i+1/2}$ and $\mathbf{F}^+_{i-1/2}$ the numerical fluxes at the interfaces.

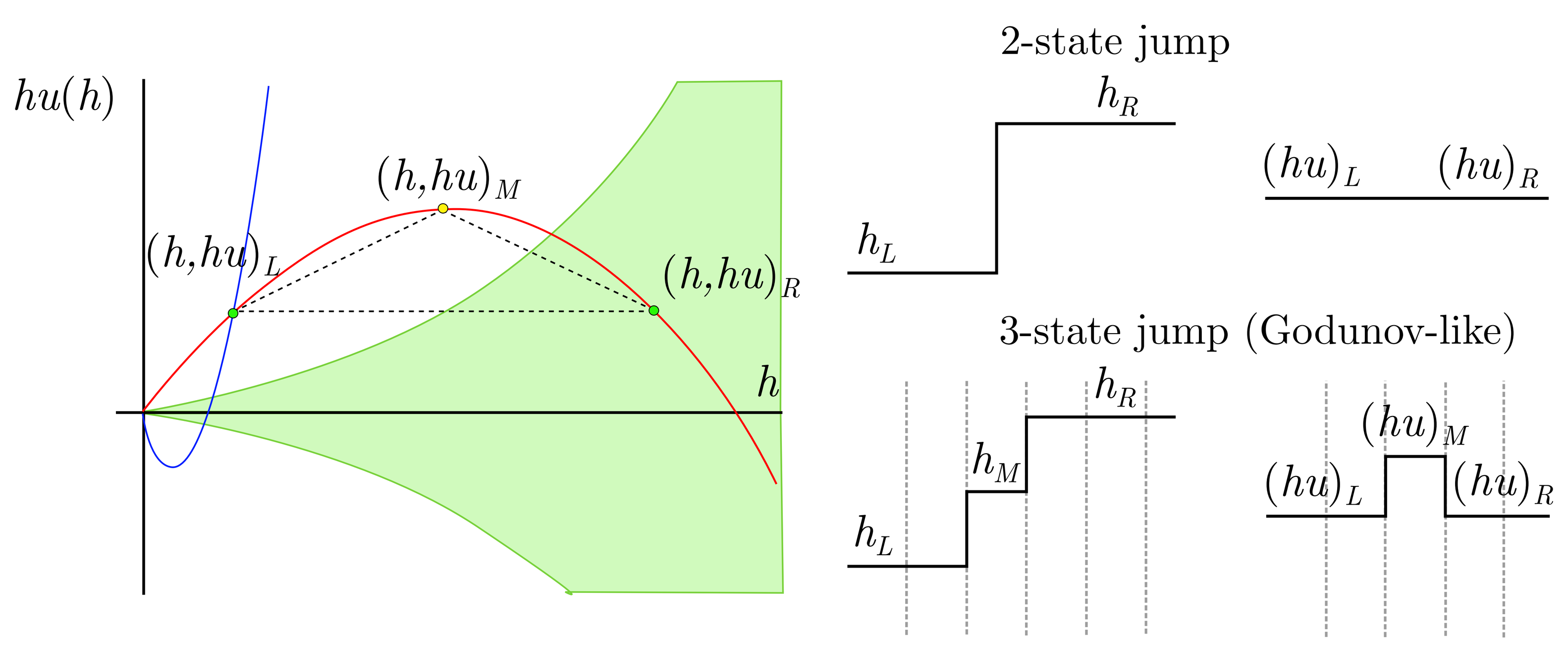


Figure 3: Hugoniot locus and schematic solution for a 2 and 3-state shocks, where the latter represents the solution provided by Godunov's scheme.

▷ Physical shockwaves have a finite width which is determined by the physical dissipation processes [4].

▷ When considering numerical shockwaves, a numerical width, usually much greater than the physical width, is enforced.

▷ This leads to the appearance of intermediate states which cannot be given a direct physical interpretation. Such states cannot be removed even when refining the grid and are the source of the slowly-moving shockwave anomaly.

4. The proposed solution: a spike reducing scheme

Given by the following steps:

1. Flux interpolation according to Zaide and Roe [4]:

$$\check{\mathbf{F}}_i = \frac{1}{2}(\mathbf{F}_{i+1} + \mathbf{F}_{i-1}) - \frac{1}{2}\tilde{\mathbf{J}}_{i-1,i+1}(\mathbf{U}_{i+1} - 2\mathbf{U}_i + \mathbf{U}_{i-1}) \,, \qquad (1)$$

2. Correction of $\check{\mathbf{F}}_i$ by computing a novel flux estimation $\hat{\mathbf{F}}_i$ to ensure exact balance between sources and fluxes in steady state [5], as follows:

$$\hat{\mathbf{F}}_i = \check{\mathbf{F}}_i - (1 - x_{\mathcal{S},i})\bar{\mathbf{S}}_{i-1,i+1} + \bar{\mathbf{S}}_{i-1/2} \qquad (2)$$

where $x_{\mathcal{S}}$ is the shock position inside the cell and $\bar{\mathbf{S}}_{i-1,i+1}$ and $\bar{\mathbf{S}}_{i-1/2}$ are suitable approximations of the source term.

3. Upwind the corrected fluxes $\hat{\mathbf{F}}_i$ at the interface to construct the numerical fluxes using an augmented solver:

$$\mathbf{F}^{-}_{i+1/2} = \hat{\mathbf{F}}_i + \sum_{m=1}^{I} [(\hat{\gamma} - \beta)\tilde{\mathbf{e}}]^m_{i+\frac{1}{2}} \,,$$
$$\mathbf{F}^{+}_{i+1/2} = \hat{\mathbf{F}}_{i+1} - \sum_{m=I+1}^{N_\lambda} [(\hat{\gamma} - \beta)\tilde{\mathbf{e}}]^m_{i+\frac{1}{2}} \,. \qquad (3)$$

where $\hat{\gamma}$ are the components of $\hat{\mathbf{\Gamma}}_{i+1/2} = \tilde{\mathbf{P}}^{-1}_{i+1/2}\delta\hat{\mathbf{F}}_{i+1/2}$, the projection of the jump in the extrapolated fluxes across cell interfaces, $\hat{\mathbf{F}}_{i+1/2} = \hat{\mathbf{F}}_{i+1} - \hat{\mathbf{F}}_i$ and β are the components of $\bar{\mathbf{B}}_{i+1/2} = \tilde{\mathbf{P}}^{-1}_{i+1/2}\bar{\mathbf{S}}_{i-1/2}$.

4. Numerical results

4.1. Steady jump over a hump

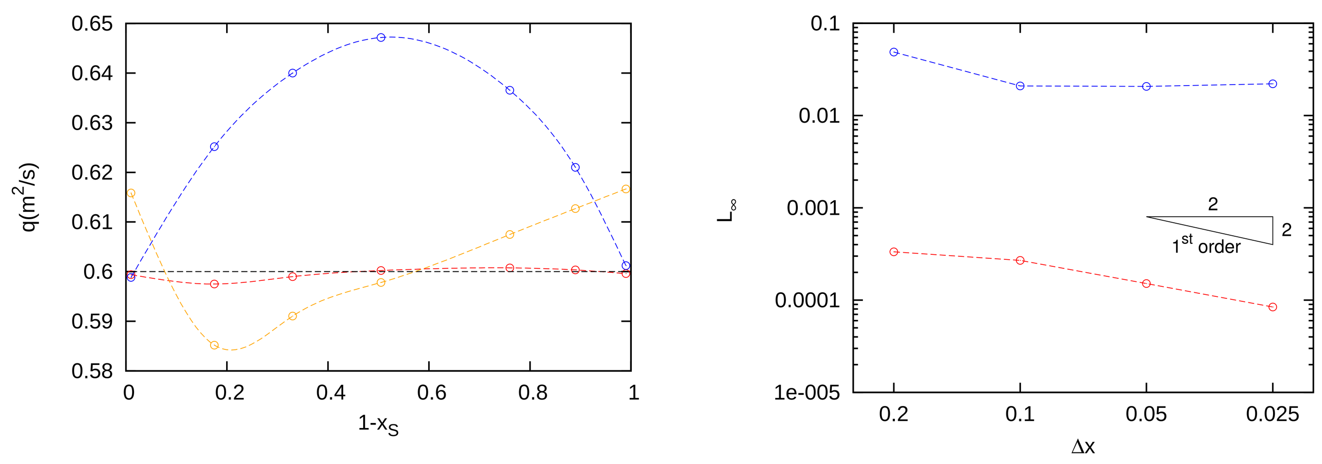
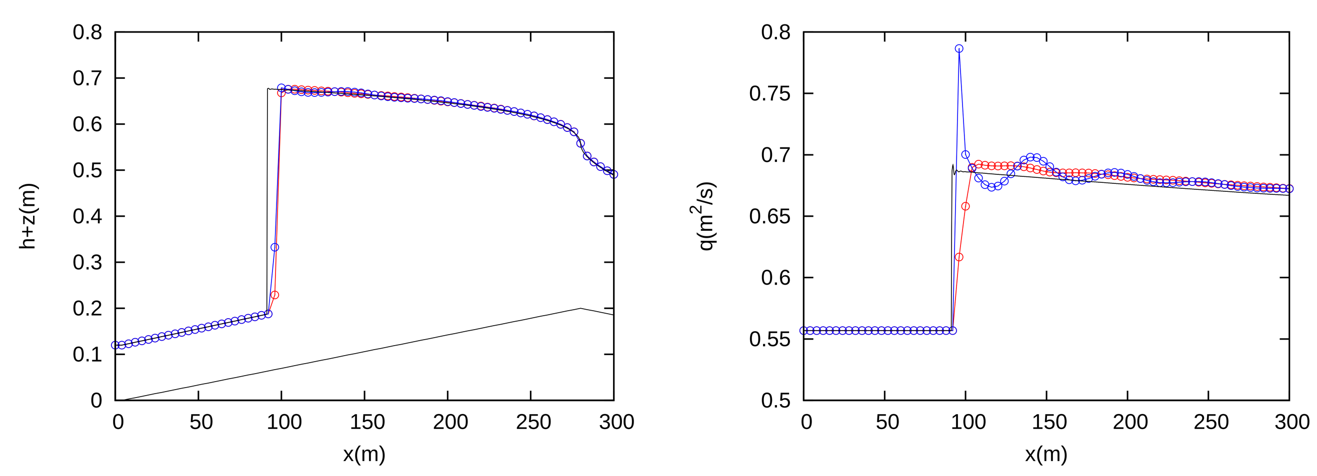


Figure 4: Test case 3. Left: representation of the spike of discharge against the position of the shock within the cell for the traditional Roe flux (– ○ –), for the method using the interpolated flux in [4] (– ○ –) and for the proposed spike-reducing method (– ○ –), using 100 cells and CFL=0.45. Right: convergence rate test for the traditional Roe method (– ○ –) and for the proposed method (– ○ –), using CFL=0.45.

2.1. Slowly-moving jumps over different bed configurations



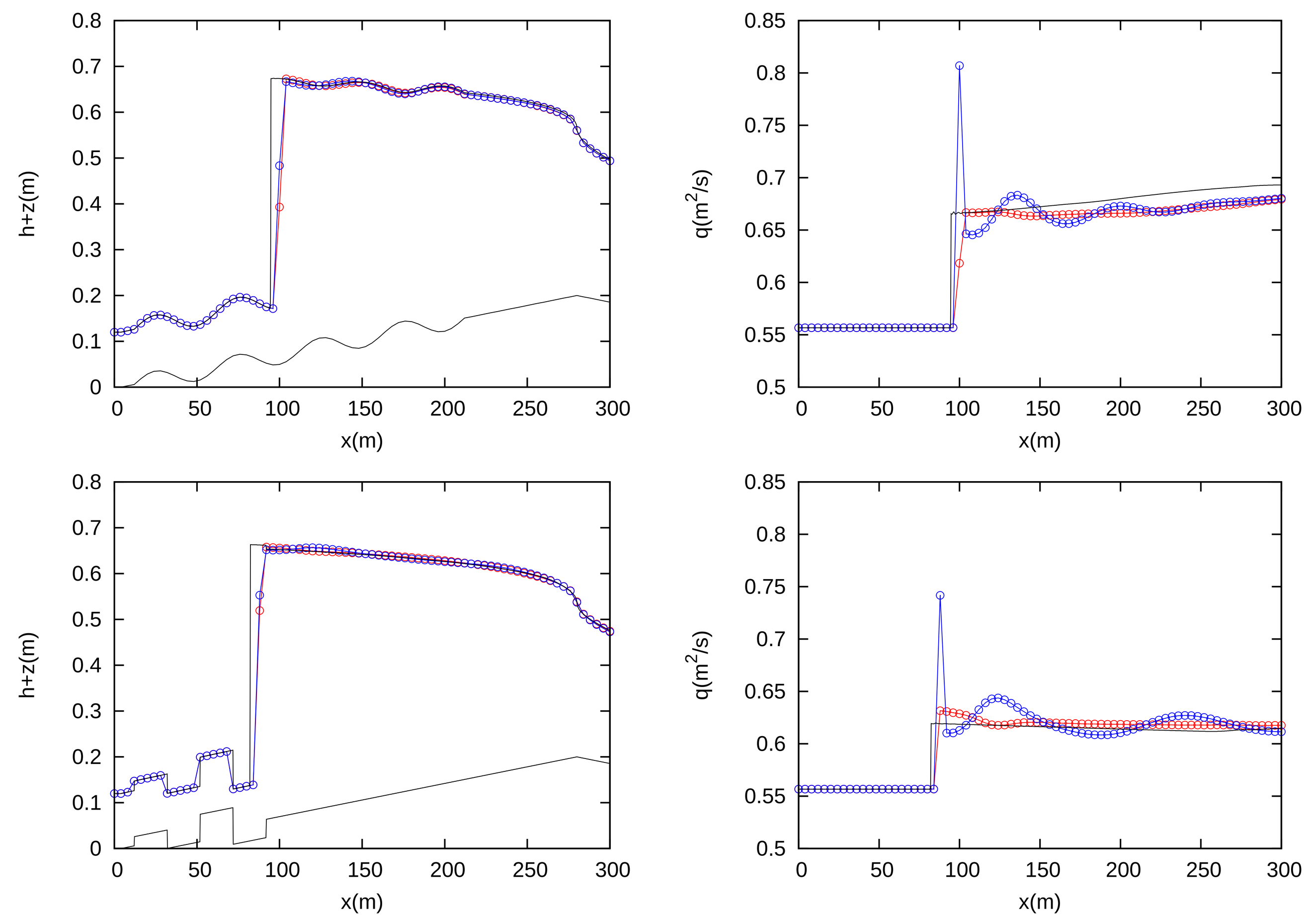


Figure 5: Test case 4. Numerical solution at $t = 610$ s for the water surface elevation (left) and discharge (right) provided by the traditional Roe flux (– ○ –) and by the proposed spike-reducing method (– ○ –), using 140 cells and CFL=0.45.

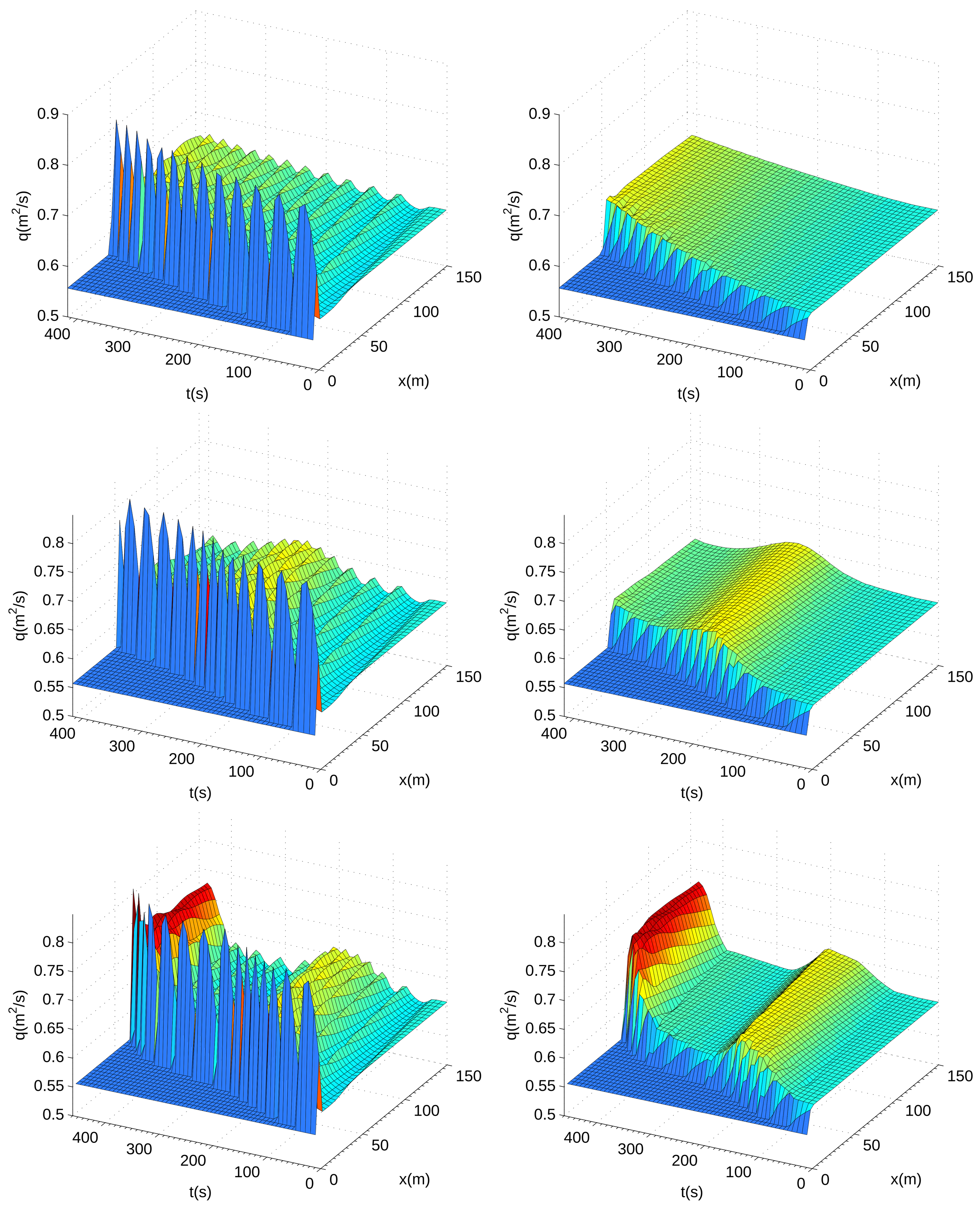


Figure 6: Test case 4. Space-time representation of the numerical discharge provided by the traditional Roe flux (left) and by the proposed spike-reducing method (right), using 140 cells and CFL=0.45.

References

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